Math 222A Lecture 1 Notes

Daniel Raban

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1 Introduction to Partial Differential Equations

We will not be strictly following a textbook, but the main outline will about the same as *Introduction to Partial Differential Equations* by Evans. The main difference will be when we cover the theory of **distributions**. For this, you could use the book by Strichartz.[1] A more in-depth reference would be the book by Friedlauder. The hardcore option is volume 1 of Hormander's books.

1.1 Basic notation for partial differential equations

Partial differential equations is the next level up from ordinary differential equations. For ODEs, we have functions $u : \mathbb{R} \to \mathbb{R}$ (or sometimes $\mathbb{C} \to \mathbb{C}$).

Definition 1.1. An **ordinary differential equation** is an equation $F(u, u', u'', \dots, u^{(n)}) = 0$, where F is a function.

Example 1.1. A **linear** ordinary differential equation is of the form $a_0 u + a_1 u' + \cdots + a_n u^{(n)} = f$, where the a_i are the **coefficients**.

Linear ODEs are generally covered in calculus, and nonlinear ODEs are generally covered in a class on ODEs. Further in this direction is the study of dynamical systems.

The above type of ODE is called a **scalar equation**.

Definition 1.2. A **system** of ordinary differential equations concerns $u : \mathbb{R} \to \mathbb{R}^n$ with

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

A partial differential equation concerns $u : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$) with $x = (x_1, \dots, x_n)$ and $\partial = (\partial_{x_1}, \dots, \partial_{x_n}) = (\partial_1, \dots, \partial_n)$. Higher order partial derivatives look like: $\partial_1^2, \partial_1\partial_2, \partial_2^2$. The solution for this notation is *multiindices*, where we denote $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} =: \partial^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a **multiindex**. In this case, the **order** is $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

[1] Strichartz is alive at 102 years old, so this book is quite old!

Definition 1.3. A **partial differential equation** looks like $F(u, \partial u, \ldots, \partial^{(m)}u) = 0$, which we may also write as $F(u^{(\leq m)}) = 0$.

Example 1.2. Linear PDEs look like

$$\sum_{|\alpha| \leq m} c_\alpha(x)\partial^\alpha u = f.$$

If $f = 0$, we call this a **homogeneous equation**, and if $f \neq 0$, we call this an **inhomogeneous equation**. where f is the **source term**.

The course will be about 80% linear PDEs and about 20% nonlinear PDEs. Nonlinear PDEs are generally harder and require knowedledge of corresponding linear PDEs to understand. The exception, which we will study in this class, is **first order scalar equations**, $F(x, u, \partial u) = 0$. This will require ODEs.

1.2 Scalar equations vs systems

Every scalar equation is already a system, but we can do better: If we have a scalar equation $F(u^{(\leq m)}) = 0$, we can convert it to a first order system. We do this by basically changing notation, writing the derivatives of u as separate functions: $u_\alpha := \partial^\alpha u$, where $|\alpha| \leq m$. This changes the equation to $F(u_{\leq m}) = 0$, where $u_{\leq m}$ refers to the collection of α-partial derivatives of u with $|\alpha| \leq m$. The functions are related by $\partial_j u_\alpha = u_{\alpha+e_j}$, where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, with a 1 in the j-th coordinate.

This conversion is not a correspondence. The class of first-order systems is much more complex than the class of scalar equations.

1.3 Important examples of linear PDEs

Example 1.3. The **transport equation** is the following linear PDE:

$$\sum_{j=1}^{n} a_j \partial_j u + bu = f.$$

Example 1.4. The **Laplace operator** Is $\Delta = \partial_1^2 + \cdots + \partial_n^2$. The **Laplace equation** is

$$-\Delta u = f.$$

Example 1.5. The **heat equation** is

$$\partial_t u - \Delta u = f.$$

Here, we think of $u(x, t)$ as a function of x and t. You may think this is just a notational convention, but in practice, it is very useful to think of one of the variables as representing time.

The heat equation describes a system evolving in time, so we call it an **evolution equation**. By contrast, the Laplace equation is a **static equation**. Notice that the Laplace equation and the heat equation are the same in the case that u is constant with respect to time. For this reason, we can interpret it as looking for the **steady states** of the evolution equation.

Example 1.6. The **wave equation** is another evolution equation, given by

$$(\partial_t^2 - \Delta)u = f.$$

The wave equation describes sound waves, electromagnetic waves, gravitational waves, elastic waves, and more.

The wave equation is an evolution equation, so it needs *initial data*. Write it as $\partial_t^2 u = \Delta u + f$. Compare this to Newton's law: $\partial_t^2 u$ corresponds to the acceleration, and $\Delta u + f$ corresponds to the force. We write our initial position as $u(t = 0, x) = u_0(x)$ and our initial velocity as $\partial_t u(t = 0, x) = u_1(x)$.

Example 1.7. The **Schrödinger equation** is a complex evolution equation of the form

$$(i\partial_t + \Delta)u = 0.$$

Here, u is a complex function. This is the fundamental equation of quantum mechanics.

1.4 What do we want to study?

In calculus, you learn methods to calculate solutions to differential equations. This is not the purpose of this course. We want to understand:

1. Existence: Does the equation have solutions?
2. Uniqueness: Is the solution the only one; i.e. does it definitively describe the behavior of the system we are studying, or can we not tell?
3. Continuous dependence on data: If we change our initial data, how does the solution change?

These three questions constitute **well-posedness theory**. We can also study:

4. Properties of solutions.